Theory of Vibration of the Larynx[1]

By R. L. WEGEL

The vibration in the larynx is caused by an automatic modulation by the vocal cords of the air stream from the lungs. Analytically the mechanism is the same, and physically, closely analogous to that of the vacuum tube oscillator. It depends principally on the resonance of the vocal cords, the modulation of air friction in the glottis by their motion and the attraction due to constriction of the air stream between them. When these forces exist in certain relative proportions and phases, sustained oscillation as in singing takes place. The whole mechanism may be represented analytically by force equations, from which conditions for accretion or subsidence of the vibration or for sustained oscillation may be easily deduced. These equations also show the analogy with other types of oscillating systems.

IT is customary in treating the theory of the voice to assume the glottis or space between the vocal cords to be a source of a steady stream of air with superimposed periodic impulses caused by the vibration of the vocal cords. The harmonic content of these impulses is modified by the "resonating" vocal cavities before being radiated into free air. It is the nature of this modification which receives most attention. The mechanism by which the vibration of the vocal cords is maintained has not been carefully studied.

The vocal cords are maintained in a state of sustained vibration by the proper balance between the various mechanical constants of the complete system, which thus act as a transformer of a part of the non-vibratory power derived from the air stream from the lungs into the vibratory power resulting in sound. It is a simple theory of this mechanism which is considered here.

The method used is to obtain the force equations, which describe the vibrations of the complete mechanical system, by means of the Lagrange equations, from expressions of the total instantaneous kinetic and potential energies, the instantaneous forces acting and rate of dissipation of energy. The resulting simultaneous equations relating to the displacements and velocities of the various parts are then studied to find the frequencies of free vibration and the relations which must obtain between the various mechanical parameters of the system in order that one of these frequencies be sustained. The method is an application of the theory of H. W. Nichols, published in *Physical Review*, August, 1917.

The theory is reduced to easily workable form by the introduction of simplifying approximations which will be described in the progress

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of the discussion. The principal one of these is the neglecting of all reactions of second or higher order, thus leaving a set of linear differential equations.

Structure of the Vocal Tract

The vocal tract consists of three principal parts, the lungs and associated respiratory muscles for maintaining a flow of air, the

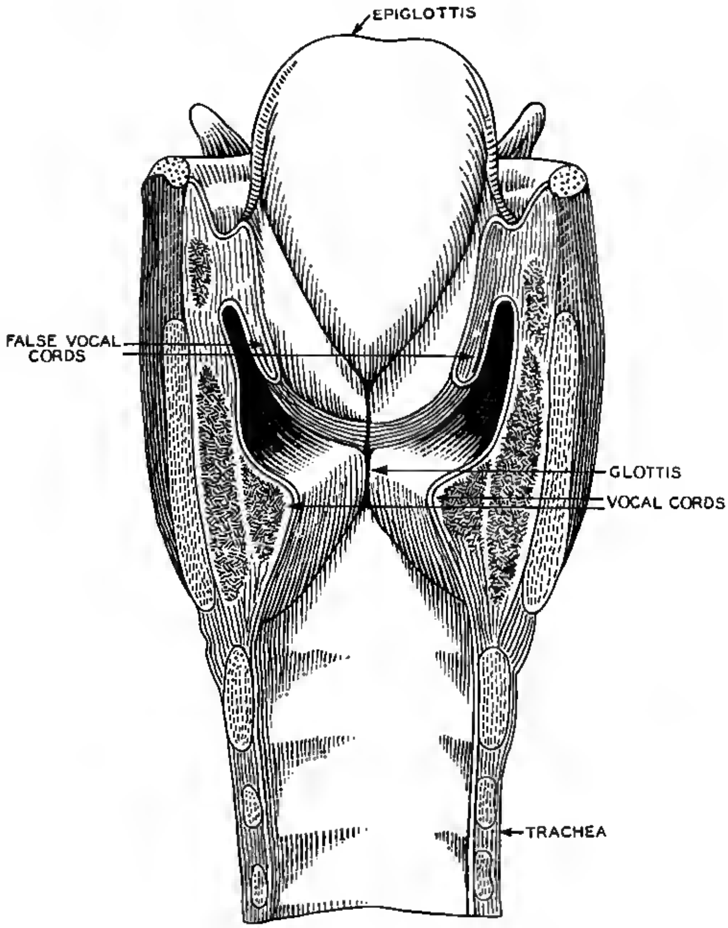


Fig. 1—Anterior-Posterior Section of the Larynx.

larynx (see Fig. 1) for producing the periodic modulation and the upper vocal cavities, pharynx, mouth and nose for varying the relative harmonic content of sound originating in the larynx.

The capacity of the lungs in an adult man is capable of being varied from about two to five liters. The average in quiet breathing is about 2.6 liters. The average expiration of air in quiet breathing is about .5 liter. The rate of expiration of air in medium loud singing varies from 40 to 200 cm.^{3}/sec., the lower values obtaining for trained singers.

The larynx (see Fig. 1) consists of an irregularly shaped cartilaginous box at the top end of a tube, the trachea, about 12 cm. long by 2 cm. in diameter, leading from the lungs. The larynx contains the vocal cords, a pair of fibrous lips which in vibrating vary the width of the slit called the glottis, between them. The length of the glottis in the adult male averages about 1.8 cm. and in the female 1.2 cm. The width of the glottis varies widely with differing sounds. A few tenths of a millimeter may be considered representative. The tension and separation of the vocal cords are controlled by muscles.

The principal upper vocal cavities are the pharynx, a space just over the larynx, the mouth and the nasal cavities. The first and second may be varied in size and shape at will, but the effect on the last is controlled only by varying the communicating aperture between it and the pharynx.

Equations of Motion of the Larynx

Fig. 2 shows a cross-section of a model which illustrates the essential details of the larynx in so far as it is necessary for this treatment. S_0 represents the area of the opening to the trachea. The vocal cords are represented by elastically hinged members of combined effective area S_2. By effective area is meant the area of aperture which displaces the same volume of air as the vocal cords when it moves the distance q_2 of the tips of the cords. This area is less than that of the vocal cords. The tips of the vocal cords are separated to form a gap, the glottis, of area S_1. A positive or up and outward displacement q_2 of the vocal cords increases S_1. It will be assumed that the air is not appreciably compressed in the neighborhood of the glottis, that is, any tendency to compression is relieved by flow into the trachea or pharynx.

The pressure in the lungs forces a steady current of air through the glottis. Let the velocity in the trachea of this steady flow be I_0 and in the glottis I_1. Small vibrations of the vocal cords superimpose additional small velocities, i_0 and i_1, in the trachea and glottis respectively. If the instantaneous velocity of the vocal cords be i_2 and it be assumed that they are constrained to move in synchronism

$$(I_0 + i_0)S_0 = (I_1 + i_1)S_1 + i_2 S_2. \tag{1}$$

The above material is a description of a simple model of two degrees of freedom which simulates the principal characteristics from the standpoint of performance of the more complex larynx which has many degrees of freedom. It is this idealized model which will be considered in the subsequent treatment. Such points of performance of the actual larynx which may be due to the action of ignored and

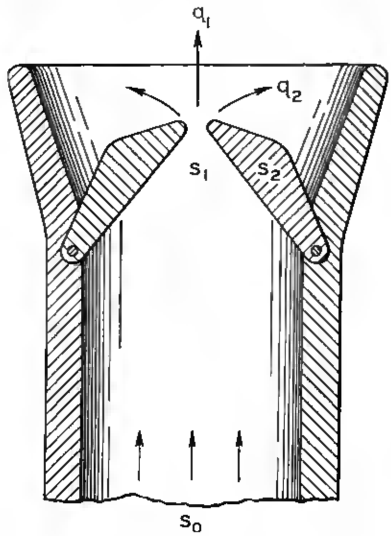


Fig. 2—Schematic Larynx Model.

presumably subsidiary modes of motion will, of course, not be predicted by the theory. These are assumed to be of minor importance. The possible independence of motion of the two vocal cords will be considered later, however.

The contraction of the air stream at the glottis introduces a relatively large concentrated kinetic energy in the air stream at this point similar to that at the mouth of a Helmholtz resonator. The inertia of a small plug of air between the vocal cords may then to a first approximation be treated as a mass L_1. A concentration of frictional resistance also occurs at this point due to viscosity and to turbulence. A positive displacement q_2 (outward) of the vocal cords causes an increase in the mass of the plug of air in the glottis and a change in the effective resistance, R, encountered by it. The inertia L_1' and resistance R of the glottis are therefore both functions of q_2, the displacement of the vocal cords from a mean position, and of the width of the glottis. If further Q_2 represent the average displacement of

the vocal cords from an appropriately chosen reference position, L_2 their inertia coefficient and K_2 their effective stiffness, all measured at the tips, the total kinetic energy, T, and potential energy, V, of the larynx are

$$T = \tfrac{1}{2}L_1'(I_1 + i_1)^2 + \tfrac{1}{2}L_2 i_2^2, \tag{2}$$

$$V = \tfrac{1}{2}K_2(Q_2 + q_2)^2. \tag{3}$$

The Lagrange equation of forces for the nth coordinate of any system is

$$E_n = \frac{\partial}{\partial t}\frac{\partial T}{\partial i_n} - \frac{\partial T}{\partial q_n} + F_n + \frac{\partial V}{\partial q_n}, \tag{4}$$

in which F_n is a reaction due to friction. The force equations for the glottis and vocal cords therefore become

$$E_0 = F_1 + \frac{\partial}{\partial t}\frac{\partial}{\partial i_1}\left[\tfrac{1}{2}L_1'(I_1 + i_1)^2\right], \tag{5}$$

$$0 = L_2\frac{di_2}{dt} + F_2 + K_2(Q_2 + q_2) - \frac{(I_1 + i_1)^2}{2}\frac{\partial L_1'}{\partial q_2}. \tag{6}$$

Nature of the "Constants" of the System

It is quite safe to conclude that none of the coefficients (inertia, dissipation and stiffness) of the larynx are sensibly constant over the range of operation of the coordinates. Direct measurements are evidently impossible. It is conceivable that they may be arrived at indirectly by means of a comparison of experimental data, especially taken for the purpose, on voice curves and the results of dynamic analysis of the kind described here. The problem may also be studied by means of models. In order to solve equations 5 and 6 it is, however, necessary to evaluate the space and velocity derivatives.

A few simple experiments were performed on models for the sole purpose of determining the qualitative nature of variation of resistance of the glottis with displacement of the vocal cords. A diagram of the model used in the measurements is shown in Fig. 3. This consists of a brass tube, *a*, 3/4″ in diameter, beveled off on the top at an angle of 45° with the axis, and two 1/8″ brass plates, *b*, fitted on these beveled surfaces so as to leave a slit, *S*, which was made adjustable in width. A cross-section of this model is shown in *c*. The bottom of the tube was attached to a large air chamber in which the pressure and velocity of air flow could be regulated and measured.

Three shapes of "glottis" were measured. The first had square corners, as shown on Fig. *3c*. The second, *3d*, was the same as *3c*,

except that the corners of the lips were rounded. The third, Fig. 3*e*, had square corners as before, but the slit was about .1 mm. wider in the middle than at the ends.

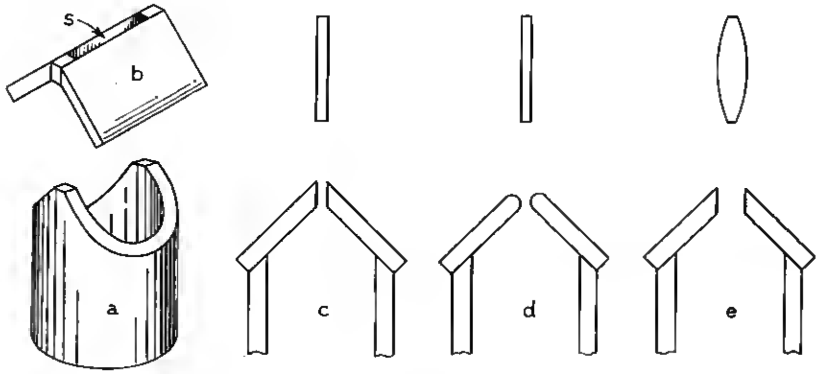


Fig. 3—Glottis Models.

The resistance R is given as the ratio of the product of pressure and slit area to the linear velocity of flow. Measurements were made in each case through a range of pressures such as to give fluxes through the slit through a range of 50 to 200 cm.^{3}/sec. (Stanley and Sheldon values, see *Sci. Am.*, Dec. 1924) and through a range of slit width W of .01 to .10 cm. The data can be represented approximately in this range for the three slits by the following formulæ:

$$R = 3.6\ I^2 W^{1.7} \times 10^{-6},$$
$$R = 6.1\ I^2 W^{2.2} \times 10^{-6},$$
$$R = 800\ I^2 W^{2.3} \times 10^{-6}.$$

In these expressions I is the velocity of flow of air through the slit. More careful data taken through a wider range of I and W would undoubtedly have given R in a power series.

These formulæ are taken to indicate that the resistance of the actual glottis increases faster than a linear function of I and W due to turbulence and may be represented as a single valued function of either displacement of the vocal cords q_2 (or glottis width) or of air velocity as expressed by a Taylor's series as follows:

$$R = R_0 + \frac{\partial_0 R}{\partial q_2} q_2 + \frac{\partial_0 R}{\partial \dot{i}_1} \dot{i}_1 + \tfrac{1}{2}\left(\frac{\partial_0{}^2 R}{\partial q_2{}^2} q_2{}^2 + \frac{2\partial_0{}^2 R}{\partial q_2 \partial \dot{i}_1} q_2 \dot{i}_1 + \frac{\partial_0{}^2 R}{\partial \dot{i}_1{}^2} \dot{i}_1{}^2\right) + \text{etc.} \quad (7)$$

In this expression R_0 is the resistance measured in the reference position at which point the derivatives are taken, where i_1 and q_2 are zero. The experiment mentioned above determines the signs of the coefficients of q_2 and i_1 as positive. If the flow were purely laminar, i.e. due to viscosity only, the first would be negative and the second zero.

The Reaction F_1

By definition, $E_0 = R_0 I_1$, where E_0 is the force of the lung pressure on the glottis and I_1 a corresponding linear velocity of flow of air. If a force F_1 slightly greater than E_0 act on the glottis and result in a velocity $I = I_1 + i_1$,

$$\frac{F_1}{I_1 + i_1} = R. \tag{8}$$

A combination of (7) and (8) constitutes an evaluation of F_1 for substitution in the force equation (5). To a first order approximation then:

$$F_1 = R_0 I_1 + \left(R_0 + I_1 \frac{\partial_0 R}{\partial i_1} \right) i_1 + I_1 \frac{\partial_0 R}{\partial q_2} q_2. \tag{9}$$

The coefficient of q_2 is dimensionally a stiffness and that of i_1 a resistance. In what follows they will be denoted by

$$F_1 = R_0 I_1 + R_1 i_1 + K_u q_2. \tag{10}$$

Glottis Mass (L_1') Reactions

The kinetic energy of the air stream being proportional to the volume integral of the square of the velocity is largely concentrated in the glottis on account of the relatively high velocity at this point. On account of the irregularity in shape and turbulence in the stream it is impracticable to attempt an integration. If the velocity were so small that the turbulence were absent an approximate value of the air mass would be obtained by taking the mass of a cylinder of air having the length of the slit and a diameter equal to its width. This would make the mass L_1' proportional to W^2, or since the width is proportional to displacement of the vocal cords, to q_2^2. Owing, however, to turbulence and other non-linearities, the mass is probably more nearly described as a tongue of air issuing from the glottis, the inertia L_1' of which varies as some power function of the width and also of the velocity.[2]

[2] It has been found since experimentally that the mass reaction is very nearly that of a cylinder as described but reduced somewhat in diameter due to viscous or turbulent drag at the tips of the vocal cords.

It might be seen by carrying through an expression for this glottis mass involving a function of velocity similar to that for R of equation (7) that only a quantitative change in effective mass would result in the final equations and that no new type of reaction would be introduced. This demonstration is not included here. In order to save space in this qualitative treatment it is ignored. For small displacements q_2 from a reference position at which the velocity of the air is I_0, the glottis inertia may be represented by the direct function:

$$L_1' = L_1 + \frac{\partial_0 L_1}{\partial q_2} q_2 + \tfrac{1}{2} \frac{\partial_0^2 L_1}{\partial q_2^2} q_2^2 + \text{etc.,} \tag{11}$$

in which the coefficient of q_2 is obviously positive. The second term of the second member of (5) may now be evaluated by performing the differentiations as indicated. Neglecting second and higher order terms and denoting dq_2/dt by i_2 the reaction in question becomes

$$L_1 \frac{di_1}{dt} + I_1 \frac{\partial_0 L_1}{\partial q_2} i_2. \tag{12}$$

The glottis mass of air, therefore, introduces two kinds of reactions: a simple inertia and a reaction proportional to the velocity of the vocal cords. For simplicity of notation (12) will be written

$$L_1 \frac{di_1}{dt} + G i_2. \tag{13}$$

This completes the evaluation of the terms (5), the force equation of the glottis, which may now be written

$$E_0 = R_0 I_1 + R_1 i_1 + K_u q_2 + \frac{L_1 di_1}{dt} + G i_2. \tag{14}$$

Force Equation of the Vocal Cords

The force equation (6) of the vocal cords contains four terms. The first is the inertia reactance of the vibrating lips. The mass L_2 is the effective vibrating mass which, if multiplied by one-half the square of the velocity at the cord tip, gives the kinetic energy of their motion. If the distribution of the velocity in the vocal cords were known this might be found by integration. The second term F_2 in equation (6) represents the internal dissipation and is assumed proportional to the small velocity i_2. The third term is the elastic reaction which is proportional to displacement.

The fourth term is a "gyrostatic" term. This term may be written as follows:

$$-\frac{\partial T}{\partial q_2} = -\tfrac{1}{2}(I_1 + i_1)^2\left(\frac{\partial_0 L_1}{\partial q_2} + \frac{\partial_0{}^2 L_1}{\partial q_2{}^2}q_2 + \text{etc.}\right). \tag{15}$$

Again by neglecting second and higher order effects this reaction becomes

$$-\frac{I_1{}^2}{2}\frac{\partial_0 L_1}{\partial q_2} - I_1\frac{\partial_0 L_1}{\partial q_2}i_1 - \frac{I_1{}^2}{2}\frac{\partial_0{}^2 L_1}{\partial q_2{}^2}q_2. \tag{16}$$

It will be seen that the first term of this expression represents a static force tending, since it is negative, to draw the vocal cords together. This is the Bernoulli effect utilized in a venturi meter. This steady force is counterbalanced by an elastic reaction of the vocal cords with which it combines to determine an equilibrium position which obtains when the cords are not vibrating. This term may, therefore, be dropped from the final equations representing only superimposed motions.

The coefficient of i_1 is identical, except for a sign, with G of (13). It represents a force on the vocal cords due to a superimposed part of the Bernoulli effect caused by the small superimposed velocity i_1 in the glottis. The coefficient of q_2 is dimensionally a stiffness. This apparent stiffness is due to the nature of the air flow and is independent of any elastic members. It is negative if the second differential of glottis mass with respect to cord displacement is negative, positive when this coefficient is positive and vanishes when this coefficient is zero. It simply adds or subtracts in effect from the stiffness K_2 of the vocal cords. The first possibility is the more likely.[3] These terms may then be written for simplicity

$$-\frac{\partial T}{\partial q_2} = -F - Gi_1 - K_x q_2. \tag{17}$$

Force Equations of the Larynx

The force equations of the glottis and vocal cords with constants thus evaluated are

$$E_0 = L_1\frac{di_1}{dt} + R_1 i_1 + Gi_2 + R_0 I_1 + K_u q_2, \tag{18}$$

$$0 = L_2\frac{di_2}{dt} + R_2 i_2 + K_2 q_2 + K_2 Q_2 - F - Gi_1 - K_x q_2. \tag{19}$$

As explained before, $E_0 = R_0 I_1$ and $F = K_2 Q_2$; so these cancel and are of no interest here. In the following it will be seen that the field

[3] This coefficient has since been found to be negative.

stiffness K_x is included in K_2 to simplify notation. This leaves (18) and (19) finally:

$$0 = L_1 \frac{di_1}{dt} + R_1 i_1 + G i_2 + K_u q_2, \tag{20}$$

$$0 = L_2 \frac{di_2}{dt} + R_2 i_2 + K_2 q_2 - G i_1. \tag{21}$$

It should be noted that these equations represent all first order internal reactions of the idealized model of the larynx. The series expansions have been carried out, to show to what approximations these equations hold. It should also be pointed out that the effects of mechanical hysteresis of the parts, which make the relative positions of the parts dependent on the previous history of their motion, have not been considered. A consideration of hysteresis complicates the theory considerably and is ignored for the same reason and with the same justification and limitations that it is ignored in the elementary treatment of electrical circuits containing coils with magnetic material and condensers with electrostatic hysteresis.

External Reactions of the Trachea and Vocal Cavities on the Larynx

So far the modifying effect of the trachea and lungs, as well as the upper vocal cavities, on the motion have not been considered. Before using the equations it is necessary to evaluate these reactions and add them in their proper places.

Imagine a weightless piston fitted into the trachea just below the vocal cords such that the volume of air thus enclosed in the larynx is so small in comparison to that of the trachea and lungs that its compressibility may be neglected. If the vocal cords are held rigid and the plug or piston of air in the glottis is forced inward, a reaction in addition to the resistance and inertia of the glottis will be encountered due to the impeding effect of the trachea piston, which impedance is determined by the constants of the lower chambers. If a small force f_0 act on the trachea, causing a small velocity, i_0, and we assume linearity of response $f_0 = Z_0 i_0$ where Z_0 is a constant which may, due to a positive inertia reactance or a stiffness, contributed by air compression in the lungs, involve either a time derivative or integral of displacement. For the present consider it to be a generalized impedance operator. Due to the relative incompressibility of the air in the larynx, the volume displaced by the trachea piston is $i_0 S_0 = i_1 S_1$. Since also the instantaneous pressure inside the larynx

is constant on all its walls, including the surface of the trachea piston $f_0/S_0 = f_1/S_1$. We then have

$$f_1 = Z_0 \frac{S_1^2}{S_0^2} i_1. \tag{22}$$

This reaction due to the trachea must be added to those of the glottis given in (20). In like manner if the effective area of the vocal cords is S_2 a reaction f_2 must be added to their force equation

$$f_2 = Z_0 \frac{S_2^2}{S_0^2} i_2. \tag{23}$$

Due to the steady component of air flow there is a static component of pressure tending to force the cords outward. This is counter to the static Bernoulli term and again, if second order effects of small quantities be neglected, serves only to alter the equilibrium position and may therefore be disregarded here.

When the glottis plug of air is displaced inward a force is exerted on the vocal cords tending to move them outward which is relieved to a certain extent by a yield of the trachea piston. This force on the vocal cords may be shown by reasoning similar to that above to be

$$Z_0 \frac{S_1 S_2}{S_0^2} i_1. \tag{24}$$

Since this part of the system is linear, the reaction between glottis and vocal cords through this channel is reciprocal so a force is exerted on the glottis when the vocal cords are displaced of

$$Z_0 \frac{S_1 S_2}{S_0^2} i_2. \tag{25}$$

It will be noticed that S_1 is a variable because of the variation in width of the glottis while vibrating. The effect of this variation in these terms is obviously second order since i_1 is small and will therefore be neglected.

The reactions of the upper cavities might be similarly added, but they are apparently relatively small and since they are at present not quantitatively known, are disregarded in the general equations because of the increased complexity. Generally, however, Z_0 may be thought of as representing the additive effects of both upper and lower chambers.

The complete force equations of the voice for small vibrations,

taking into account all major external as well as internal reactions, may then be written:

$$0 = L_1 \frac{di_1}{dt} + R_1 i_1 + G i_2 + K_u q_2 + \frac{Z_0 S_1^2}{S_0^2} i_1 + \frac{Z_0 S_1 S_2}{S_0^2} i_2. \qquad (26)$$

$$0 = L_2 \frac{di_2}{dt} + R_2 i_2 + K_2 q_2 - G i_1 + \frac{Z_0 S_2^2}{S_0^2} i_2 + \frac{Z_0 S_1 S_2}{S_0^2} i_1. \qquad (27)$$

These equations may be put in a somewhat simpler form by virtue of the fact that they are linear differential equations with constant coefficients. In such a case the time differential may be replaced by an algebraic operator p such that $i = pq$, $di/dt = p^2 q$, where p is of the dimensions and nature of a frequency

$$0 = \left(p^2 L_1 + pR_1 + pZ_0 \frac{S_1^2}{S_0^2} \right) q_1 + \left(pG + pZ_0 \frac{S_1 S_2}{S_0^2} + K_u \right) q_2, \qquad (28)$$

$$0 = \left(-pG + pZ_0 \frac{S_1 S_2}{S_0^2} \right) q_1 + \left(p^2 L_2 + pR_2 + K_2 + pZ_0 \frac{S_2^2}{S_0^2} \right) q_2. \qquad (29)$$

The determinant of this system is (calling $pZ_0 = Y_0$)

$$D = \begin{vmatrix} \left(p^2 L_1 + pR_1 + Y_0 \frac{S_1^2}{S_0^2} \right) & \left(pG + Y_0 \frac{S_1 S_2}{S_0^2} + K_u \right) \\ \left(-pG + Y_0 \frac{S_1 S_2}{S_0^2} \right) & \left(p^2 L_2 + pR_2 + K_2 + Y_0 \frac{S_2^2}{S_0^2} \right) \end{vmatrix}. \qquad (30)$$

This determinant represents the complete reactions of the larynx and the external effects of communicating air chambers.

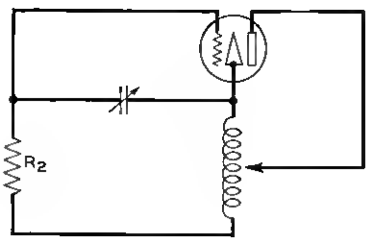
If the effects of the air chambers be disregarded the system is represented by placing $Y_0 = 0$, giving the simple form

$$D = \begin{vmatrix} (p^2 L_1 + pR_1) & (pG + K_u) \\ -pG & (p^2 L_2 + pR_2 + K_2) \end{vmatrix}. \qquad (31)$$

Nature of This System

The voice system represented by determinant (3) is very closely analogous to other vibrators, such as the microphone oscillator or door buzzer and the vacuum tube oscillator. The literature on the latter subject is now so extensive that the pointing out of the analogy should make the method of solution for sustained oscillation, as in singing, or for subsidence or accretion of the oscillation, as in speaking, clear to any one familiar with it.

Fig. 4*a* is a schematic diagram of a three-element vacuum tube oscillator circuit known as the "tuned grid" circuit. This is one of many kinds. The transformer coupling between the plate and grid circuit is represented by an auto-transformer. Fig. 4*b* represents the same circuit schematically but with circuit elements only. In this R_2 represents that part of the resistance of the coil which belongs to the grid circuit and any other associated dissipation, L_2 is the inductance of the coil as seen from the grid mesh and K_2 the reciprocal of the combined tuning capacity across the grid and that of the grid-filament. It is the electrical stiffness or elasticity of the grid mesh,



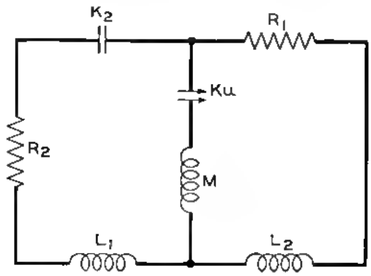


Fig. 4—"Tuned Grid" Oscillator.

in other words. R_1 is a plate-filament resistance ("a.c.") and L_1 that part of the coil in the plate circuit. M is the mutual inductance of the transformer which is not part of the mesh impedance of either plate or grid. The element K_u is the "uni-lateral mutual impedance" (G. A. Campbell, 1914) between the plate and grid meshes and is numerically equal to μK_2, where μ is the amplification constant of the tube. Other internal tube impedances are as usual neglected. The impedance determinant may be written directly from the circuit diagram Fig. 3*b*.

$$D = \begin{vmatrix} (p^2L_1 + pR_1) & (p^2M + K_u) \\ p^2M & (p^2L_2 + pR_1 + K_2) \end{vmatrix} . \tag{32}$$

The quantities on the principal diagonal of this determinant, that is the first and last elements, are as usual in a circuit determinant the mesh impedances while the others are the mutuals. The principal features of the analogy may be seen by comparison of determinants (31) and (32). Except for the thus far undefined external or trachea impedance the mesh impedances are the same, from which it appears that the glottis is analogous with the plate-filament path in the vacuum tube and the vocal cords with the grid-filament path. The air

velocity in the glottis I_1 corresponds to the plate current. In the vacuum tube this plate current is modulated by varying charge, q_2, on the grid. In the larynx the glottis air velocity is modulated by varying displacement, q_2, of the vocal cords. The charge on the plate (again neglecting internal capacities except the grid-filament) causes no effect on the grid mesh and in the larynx the position of any element of glottis air has no effect on the vocal cords. The unilateral mutual impedance, K_u, is the same in both.

The analogy breaks down at the point where the "feed back" part of the mechanisms is compared. The "feed back" is the bilateral part of the mutual impedance between the two meshes. In the vacuum tube circuit this is p^2M, the mutual of the transformer, while in the larynx it is pG, the "gyrostatic" mutual. The latter is a type of element which does not occur in electrical circuits, arising as it

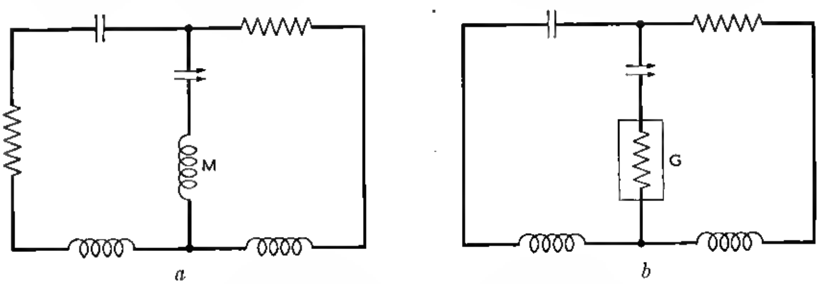


Fig. 5—Tuned Grid and Wind Reed Circuits.

does from a variation of a mass or inductance with a displacement. Inductance, being a function purely of the geometry of a circuit, can only vary with mechanical displacement and not with electrical displacement or charge. The gyrostatic mutual is common in the mechanics of rotating bodies whence it derives its name. It is also the mutual in an electromagnetic telephone receiver or relay between the electrical circuit and the armature or diaphragm.

In order to fix the rather useful concept of the analogy in mind, Fig. 5 is added showing the schematic circuit of the vacuum tube (5*a*) and a circuit diagram (5*b*), which represents determinant (31) the characteristic formulation of the dynamics of the larynx. Fig. 5*b* is represented by the conventions of an electrical circuit, except for the element G for which a different convention is necessary. The one taken here is that of a resistance enclosed in a rectangle. From (31) it will be seen to be similar to a resistance in its association with frequency p but different from resistance in that it occurs non-symmetrically in sign in the determinant. It does not involve dissipation.

Its occurrence here is the simplest possible for when there are appreciable concealed or ignored modes of motion it may have the form of a generalized impedance containing at least one element of resistance, but will always be non-symmetrical as a whole in sign in the determinant.

The use of the circuit for representing the mechanical system is an extension of an old but recently popularized method of studying mechanical or electrical vibrating systems by the help of analogy, one with the other. The extension consists in the explicit representation by diagram of the gyrostatic mutual which makes the determinant unsymmetrical in sign and of the unilateral mutual which makes the determinant unsymmetrical in magnitude. Fig. 6 is a

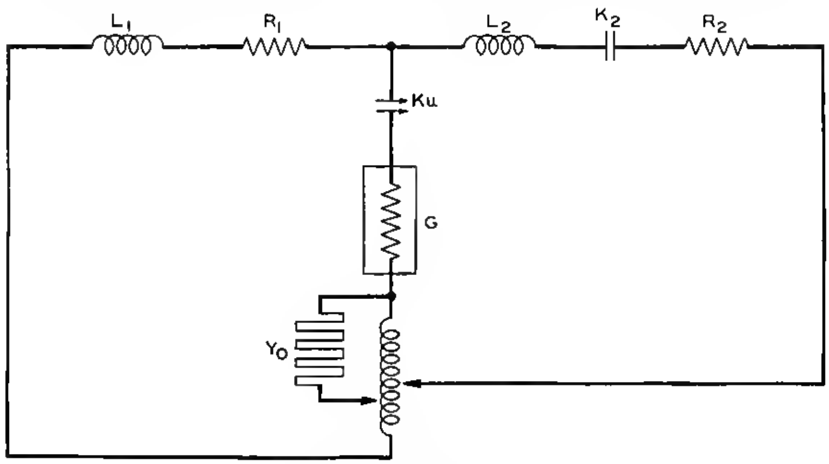


Fig. 6—General Wind Reed Circuit.

diagrammatic representation of the more general system of determinant (30). This includes the external Y_0 reactions as well as the internal.

Having thus described the extended method of analogy the following study of the larynx with the help of the circuit diagram of its determinant should be clear.

Sustained Vibration of the Simple Larynx

In vibrating, the vocal cords do not receive excitation of the frequency at which they vibrate. The source of power is in the air stream I_1 which enters the equations in K_u, the unilateral mutual impedance. Since this is treated as a constant circuit or dynamical element this air stream may be ignored as a drive and the resulting

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vibration considered as the free oscillation of the system. The determinant (31) (or 30) is then used to determine the free frequencies and decrements of the system. The method is as usual to solve for p in the equation

$$D = 0. \tag{33}$$

To simplify the demonstration the simple larynx without the load of the air chambers will be considered. Taking D of (31) then and expanding:

$$p^4 L_1 L_2 + p^3(L_1 R_2 + L_2 R_1) + p^2(L_1 K_2 + R_1 R_2 + G^2) + p(R_1 K_2 + G K_u) = 0. \tag{34}$$

If this be divided by $L_1 L_2$ and the uncoupled decrements and natural frequency defined:

$$\frac{R_1}{2L_1} = \Delta_1; \quad \frac{R_2}{2L_2} = \Delta_2; \quad \frac{K_2}{L_2} = \omega_2^2, \tag{35}$$

then

$$p\left[p^3 + p^2(2\Delta_1 + 2\Delta_2) + p\left(\omega_2^2 + 4\Delta_1\Delta_2 + \frac{G^2}{L_1 L_2} \right) + \left(2\Delta_1\omega_2^2 + \frac{G K_u}{L_1 L_2} \right) \right] = 0. \tag{36}$$

One of the roots of this equation is zero and another is negative real since all coefficients are positive. This root is therefore the decrement of a mode of non-vibratory motion. The remaining two roots may be real, imaginary or generally complex, of the form

$$\Delta \pm j\omega. \tag{37}$$

If it is found that $\Delta = 0$, then an oscillation once started will be sustained. If Δ be negative then any existing oscillation must subside or if Δ be found positive then an impulse will start an oscillation which of itself increases in amplitude to a point where its violence modifies the constants to such an extent as to make Δ vanish, leaving a sustained oscillation, or negative leaving the oscillation to subside to a lower amplitude or completely if sufficient permanent changes have been made.

If now (36) be written

$$Ap^3 + Bp^2 + Cp + D = 0 \tag{38}$$

and the first root (37) be substituted, two equations result, one from the real and the other from the imaginary terms, as follows:

$$A\Delta(\Delta^2 - 3\omega^2) + B(\Delta^2 - \omega^2) + C\Delta + D = 0, \tag{39}$$

$$A(3\Delta^2 - \omega^2) + B(2\Delta) + C = 0. \tag{40}$$

Now the condition for sustained oscillation is that $\Delta = 0$ and if the value of ω when this obtains be ω_0 then

$$\omega_0^2 = \frac{D}{B} \qquad \text{and} \qquad \omega_0^2 = \frac{C}{A}, \tag{41}$$

or the condition for sustained vibration in terms of the constants is

$$AD = BC. \tag{42}$$

In addition to this if use be made of the fact that in an algebraic equation such as (36) the coefficient of p^2 is the negative sum of all the roots then this coefficient is the real root. Let this be Δ_0 and then (36) may be written

$$p^3 + p^2\Delta_0 + p\omega_0^2 + \Delta_0\omega_0^2 = 0. \tag{43}$$

The coefficient of p in (36) is therefore the square of radial frequency at which sustained oscillation will take place and this is seen to be higher than the natural frequency ω_2 of the vocal cords, the difference being increased when the damping of either mesh is greater or when the coupling mutual G is greater.

It might be noted in passing that (43) is the free oscillation equation for any system which may be represented by a cubic equation and is not confined to the simple larynx. Such an equation always results when there is only one kind of reactive element in one of the meshes. It holds also for the tuned grid circuit.

The condition for sustained oscillation to be fulfilled for the constants may from (42) be reduced to:

$$R_1R_2 = G^2\left(\frac{L_1K_u}{GR_1} - 1\right). \tag{44}$$

It is rather difficult to place a simple physical interpretation on this formula. The qualitative import of it may however be seen by substituting the values of G and K_u from (13) and (10):

$$R_1R_2 = L_1^2I_1^2\left[\frac{\partial_0 R/\partial q_2}{R_1} - \frac{\partial_0 L_1/\partial q_2}{L_1}\right]\frac{\partial_0 L_1/\partial q_2}{L_1}. \tag{45}$$

The first term in brackets is in the nature of a resistance modulation constant, a fractional change in glottis resistance per unit displacement of the cords, to be designated by r and the second term similarly a glottis mass modulation constant, l. The quantity L_1I_1 is the momentum of the air in the glottis. This equation is then

$$R_1R_2 = (L_1I_1)^2(r - l)l. \tag{46}$$

Thus it appears that the resistance modulation must always be greater than the mass modulation and when the difference is small the air momentum must be increased to compensate. Owing to the physical limitation in accuracy of continuous maintenance of adjustment in the larynx, if a large momentum is depended upon to compensate for a small modulation difference, an unsteadiness or instability is likely to result. It is common experience that it is impossible to produce a sound with the voice with less than a certain minimum intensity. This corresponds, with the most favorable adjustment of the modulation constants which are physically possible, to a minimum momentum of air from the lungs which satisfies (46). It will be evident that this interpretation must not be taken too seriously quantitatively.

Subsidence and Accretion of Vibration of the Simple Larynx

Oscillograms made of the speaking voice show that, among other things, the amplitude of the oscillation and the pitch are in a continuous state of change. This is also true in singing but not nearly to the same extent. It seems therefore that in singing the adjustment of the voice system for sustained oscillation as described in (44) above is of major importance, while in speaking conditions for variation are of most importance.

The principle of the investigation of variation is simple enough but in all but the most elementary systems the algebra involved is impracticably awkward. If by solving (33) directly for the roots of p, it be found that Δ is positive, then any existing vibration will tend to increase while if Δ is negative, then vibration will tend to subside. The algebraic difficulties arise in the general solution but these are largely obviated by making the assumption, which is most likely usually fulfilled in practice, that the real parts of the roots may be treated as small quantities when compared with the imaginary parts. A common frequency for a man's voice is 150 cycles per second for which ω_0 is 1000 in round numbers. The decrement of a telephone receiver is ordinarily 100 to 200 in open air. The decrement of a tuning fork is represented by a fraction. Judging from variations in amplitude in an oscillogram (from which of course decrements may not be read directly) it would seem reasonable to assume that Δ is small compared with ω_0. The study of variation thus becomes an investigation of small departures from a condition of sustained oscillation, the reference condition being that critical adjustment for which the roots of interest of (33) are pure imaginary.

Suppose in (38) that $A = 1$; then without loss in generality:

$$p^3 + Bp^2 + Cp + D = 0. \tag{47}$$

In such an equation the roots are continuous functions of the coefficients. The same is true of their derivatives except at the one point where transition occurs from pure real to complex values. The values of the roots of interest in this connection are in their complex region at the point where the real part of the root passes through a zero value. This is the point at which free oscillation of the oscillating mode occurs, the values of the roots of this mode being as shown before, $\pm j\omega_0$.

If it now be supposed that one cause produces small variations, directly or indirectly on each of the coefficients and that the magnitude of this cause be x, then:

$$(3p^2 + 2Bp + C)\frac{dp}{dx} + p^2\frac{dB}{dx} + p\frac{dC}{dx} + \frac{dD}{dx} = 0. \tag{48}$$

The problem then is to determine dp resulting from any assigned cause dx when $p = j\omega_0$. From (43) we have at this point $B = \Delta_0$, $C = \omega_0^2$ and $D = \Delta_0\omega_0^2$.

$$2\omega_0^2\left(1 - j\frac{\Delta_0}{\omega_0}\right)\frac{dp}{dx} = -\omega_0^2\frac{dB}{dx} + j\omega_0\frac{dC}{dx} + \frac{dD}{dx}\,. \tag{49}$$

This is the frequency (complex) variation equation taken in the neighborhood of free oscillation.

When any readjustment of the larynx takes place all of the "constants" entering the coefficients undergo change, in particular those of the glottis K_u, R_1, G. Suppose for simplicity that one only varies, then this variation dK_u, dR_1, or dG may be taken as the magnitude of the cause dx. In particular if K_u vary,

$$dB = 0 = dC \qquad \text{and} \qquad dD/dx = G/L_1L_2,$$

$$\left(1 - j\frac{\Delta_0}{\omega_0}\right)\frac{dp}{dK_u} = \frac{G}{2\omega_0^2L_1L_2}\,. \tag{50}$$

If in addition Δ_0 be small compared with ω_0,

$$dp = \frac{GdK_u}{2\omega_0^2L_1L_2}\left(1 + j\frac{\Delta_0}{\omega_0}\right)\,. \tag{51}$$

This shows that if a condition of sustained oscillation is departed from by slightly increasing K_u, an increase in the amplitude of vibration begins which is proportional to the logarithm, since $(p + dp)$

is the exponent, of the increment dK_u and the frequency (imaginary part) of vibration increases slightly in proportion. If K_u were the only varying element the vibration would continue indefinitely to increase.

If on the other hand K_u be assumed constant, the variation being in R_1, then it may be similarly shown that

$$dp = \frac{d\Delta_1}{2\omega_0^2}\left[-\left(4\Delta_1 + \frac{G^2}{L_1L_2} + \Delta_0^2\right) + j\frac{\Delta_0\omega_2^2}{\omega_0^2}\right], \tag{52}$$

whence it appears that a small increase in glottis resistance dR_1 (or $d\Delta_1$) introduces a subsidence of vibration but an increase in frequency of oscillation as before. A decrease $-dR_1$ of course produces the opposite effect.

If the change be in G, it turns out that

$$dp = \frac{dG}{2\omega_0^2L_1L_2}\left[\left(K_u - 2G\Delta_0\right) + j\left(\frac{K_u\Delta_0}{\omega_0} + 2\omega_0G\right)\right]\cdot \tag{53}$$

Here it appears that an increase in the gyrostatic mutual, G, may introduce either a subsidence or an accretion in amplitude but like the others makes for an increase in frequency of oscillation.

Variation in other elements produces similar conflicting tendencies not only in damping but in frequency.

The physical picture to be drawn from this is that in speaking the voice modulates from one amplitude and frequency to another by proper relative variations in adjustments in its constants, being constantly in a state of changing subsidence or accretion. It would seem also that the principal cause of change in frequency is in the vocal cords and that of amplitude variation in the glottis. Speaking is, in this respect, a more intricate process than singing.

Other Types of "Feed Back"

The detailed study of the larynx has so far been limited to the assumption that the "feed back" is entirely gyrostatic. This is of course actually not the case. How much influence is exerted by the general Y_0 is difficult to estimate.

If the trachea were a long tube but still shorter than a quarter wave-length of sound at the frequency of oscillation and rather smaller in diameter, and substantially open at the end the mass of the air in it would then be appreciable and Y_0 in (30) would be written p^2L_0. If in addition the gyrostatic term were negligible the system would then be exactly analogous with the tuned grid system and (32) rather than (31) should be the subject of detailed study.

If on the other hand the lungs acted substantially as a solid walled chamber of comparatively small size, the elasticity of the contained air would be represented by taking K_0 for Y_0. The surface area in the lungs is very large compared with a regular chamber of equal volume so considerable dissipation must be encountered by vibration. If this were the most important reaction Y_0 should have been replaced by pR_0.

Unquestionably all three types of reaction enter. A more general treatment to include them is plainly not a subject for a short paper. It is interesting however to note that in the dynamical system of brass horns these latter Y_0 reactions exert controlling influences. In this case the lips of the player perform the same function as do the vocal cords of the voice while the external load, the horn, corresponds to the pharynx, the reaction of which is the same dynamically as the trachea. In this case the frequency of the horn is that of sustained oscillation and not that of the lips. The same is true of the wood-wind, in which case the reed or reeds replace the lips or vocal cords. In these cases Y_0 is proportional inversely to the hyperbolic tangent of the frequency or may be approximately represented by the impedance of an anti-resonant element.